Parametric Excitation

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This paper outlines a theory of excitation of oscillations under the rather slow variations of a parameter on which a system depends. The basic equation is a Mathieu equation. The basic method consists in reducing the system to polar coordinates and applying perturbations.

1. INTRODUCTION

IT has been known for a long time that if a parameter of an oscillatory system (electrical or mechanical) is varied periodically between certain limits, the system becomes "excited," that is, starts oscillating with frequency equal to one-half of that with which the parameter varies. The term "parametric excitation" (p.e. for short) is frequently used to designate this phenomenon.

Perhaps the best known example of p.e. is the operation of a swing, that is, the "excitation" of a swing due to a properly timed bending of the knees followed by a subsequent straightening out on the part of the person on the swing. Owing to this periodic raising or lowering of the center of gravity of the body, the motion of the swing starts from a small angle and is subsequently maintained at a considerable angle of oscillation.

Melde[1] and Lord Rayleigh[2] were first to observe and analyze this phenomenon. Rayleigh's experimental arrangement consisted of a stretched wire, one end of which was fastened to the prong of a tuning fork. When the fork is set in oscillation, thus causing a variation of tension in the string, lateral oscillations of the latter are excited with a frequency equal to one-half of that of the tuning fork.

In later years L. Mandelstam and N. Papalexi[3] constructed an interesting "parametric generator" consisting of an oscillatory circuit with a variable parameter (either the capacity C or inductance L). It is observed that when the parameter is made to vary around its average value with frequency f, the circuit begins to oscillate with frequency $f/2$ in the *absence of any external source of electric energy*. The experiment is particularly well defined when the frequency $f/2$ coincides with the natural frequency of the oscillating circuit. In their early experiments these authors oper-
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ated with a nearly linear circuit, in which case it was observed that the voltage generated in the apparatus rapidly reaches such a high value that the machine is punctured. In order to obviate this, a nonlinear resistor was inserted in series with the circuit; and it was found that the voltage (and hence the energy generated in the device) approaches a definite stationary value.

The study of this phenomenon is obviously amenable to the differential equation (d.e. for short) with periodic coefficients, a subject on which there exists a voluminous literature for the last seventy years or so. It is well known that a typical equation of this kind, the so-called Mathieu equation, has stable and unstable zones in its "parameter space." The p.e., by its nature, is to be expected in the latter. The experiment shows, however, that the phenomenon always adjusts itself so as to be in these unstable zones. There is another complication, namely, the lack of any theory of these d.e. in the nonlinear range. On the other hand, most frequently these phenomena occur precisely when the system is nonlinear. In what follows we propose to outline a theory of this phenomenon on the basis of the
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perturbation method introducing the polar coordinates $[\rho(t), \theta(t)]$ where $\rho(t)$ will be connected with the total energy and $\theta(t)$ will be the phase angle of the motion. It will be shown that the existence of this effect is closely related to the question of *stability of the phase* around such a value, for which a steady increase of energy is possible. Moreover, a nonlinear extension of this method is also relatively simple.

2. LINEAR CASE; STABILITY OF THE PHASE

Since the reduction of the various practical cases of p.e. to the Mathieu equation is well known,[4] we shall start from the d.e. of the form

$$\ddot{x}+(1+\gamma\cos\omega t)x=0, \qquad (2.1)^*$$

where x is the variable of the problem (e.g., coordinate, voltage, etc.), γ is the so-called index of modulation, and ω is the frequency with which the parameter is varied.

Equation (2.1) is equivalent to the system of

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[1] F. E. Melde, Pogg. Annalen **108**, 508 (1859).

[2] Lord Rayleigh, "On maintained vibrations," Phil. Mag. **15**, 229–235, Fifth Series (April, 1883).

[3] L. Mandelstam, N. Papalexi, A. Andronov, A. Witt, and S. Chaikin, "Exposé des recherches recentes sur les oscillations non-linéaires," J. Tech. Phys. (U.S.S.R.) **2**, 81–134 (1934) (a bibliography on parametric excitation is appended to this paper).

[4] N. W. McLachlan, *Theory and Applications of Mathieu Functions* (Oxford University Press, 1947), Sec. 15.50–15.54.

* It is preferable to deal throughout with dimensionless quantities. If one designates time by t', angular velocity by ω', the usual form of the Mathieu equation encountered in applications is $d^2x/dt'^2+(\alpha'+\beta'\cos\omega't')x=0$. Here α' and β' are obviously of dimension T^{-2}. Change of the independent variable from t' to $t=(\alpha')^{\frac{1}{2}}\cdot t'$ reduces this equation to (2.1), where $\gamma=\beta'/\alpha'$, $\omega=\omega'(\alpha')^{\frac{1}{2}}$ and $t=(\alpha')^{\frac{1}{2}}\cdot t'$ are clearly dimensionless. In Eq. (7.1) we use as independent variable t' (time), but beginning with (7.2) we deal again with the dimensionless time $t=\omega_0t'$.

equations

$$\dot{x}=y$$
$$\dot{y}=-(1+\gamma\cos\omega t)x.$$

Changing to polar coordinates, that is, setting

$$x=r\cos\theta \qquad \rho=r^2=x^2+y^2$$
$$y=r\sin\theta \qquad \theta=\arctan(y/x),$$

the above system becomes

$$d\rho/dt=-\gamma\rho\cos\omega t\sin2\theta, \tag{2.2}$$

$$d\theta/dt=-1-\gamma\cos\omega t\cos^2\theta. \tag{2.3}$$

The variable $\rho=x^2+\dot{x}^2=x^2+y^2$ may be regarded as *the square of the* solution $r(\theta)$ in the phase plane (r,θ). For a conservative system one can identify the curve $\rho(\theta)$ with the total energy of the system which is then constant. However, this does not apply here, since the system is not conservative. One may retain a certain connection with energy if one agrees to consider the motion of the representative point on the $\rho(t)$ curve as evolution of energy of the system in time. Thus, for example, if $\rho(t)\to\infty$ when $t\to\infty$, this will mean that the energy grows beyond any bound. If, however, $\rho(t)$ describes a certain closed curve $\rho(\theta)$, this will mean that a certain stationary (not necessarily constant in the instantaneous value), energy content of the system is ultimately reached. The case $\omega=2$ is of primary interest, since precisely in this case p.e. has been observed experimentally. This gives

$$d\rho/dt=-\gamma\rho\cos2t\sin2\theta \tag{2.4}$$

$$d\theta/dt=-1-\gamma\cos2t\cos^2\theta. \tag{2.5}$$

Introducing the variable $P(t)=\log\rho(t)$ and choosing any arbitrary instant of motion as $t=0$, one can set $P(0)=P_0$; $\theta(0)=\varphi_0$. The solutions of Eqs. (2.4) and (2.5) are determined uniquely and, since they depend analytically on the parameter γ, one can represent them by series development:

$$P(t)=\sum_{\nu=0}^{\infty}\gamma^\nu P_\nu(t);\quad \theta(t)=\sum_{\nu=0}^{\infty}\gamma^\nu\theta_\nu(t). \tag{2.6}$$

If one introduces these series into Eqs. (2.4) and (2.5) and equates the terms with equal powers of γ, one obtains a series of d.e.,

$$\dot{P}_0=0,\quad \dot{P}_1=-\cos2t\sin2\theta_0;$$
$$\dot{P}_2=-2\cos2t\cos2\theta_0\cdot\theta_1; \tag{2.7}$$

$$\dot{\theta}_0=-1;\quad \dot{\theta}_1=-\tfrac{1}{2}\cos2t.\ (1+\cos2\theta_0);$$
$$\dot{\theta}_2=\cos2t.\ \sin2\theta_0.\ \theta_1. \tag{2.8}$$

From these one can compute recurrently all terms of the expansion (2.6) with the initial conditions implied by the set-up, *viz.*:

$$P_\nu(0)=0;\quad \theta_\nu(0)=0,\quad \nu=1,2,\cdots. \tag{2.9}$$

In this manner one obtains

$$P_0(t)=P_0;\quad P_1(t)=-(\tfrac{1}{2}\sin2\varphi_0)t$$
$$+\text{periodic terms}\cdots \tag{2.10}$$

$$\theta_0(t)=\varphi_0-t;\quad \theta_1(t)=-(\tfrac{1}{4}\cos2\varphi_0)t$$
$$+\text{periodic terms},\cdots, \tag{2.11}$$

where the periodic terms resume the same values if t is increased by 2π.

Thus, starting from a point (P_0,φ_0) at $t=0$ in the phase plane, one reaches at $t=2\pi$ the point $[P(2\pi),\theta(2\pi)]$ given by the expressions

$$P(2\pi)=P_0-\pi\gamma\sin2\varphi_0+0(\gamma^2) \tag{2.12}$$

$$\theta(2\pi)=\varphi_0-2\pi-\tfrac{1}{2}\pi\gamma\cos2\varphi_0+0(\gamma^2), \tag{2.13}$$

where the terms $(0(\gamma^2)$ are analytic functions of φ_0 and γ and are independent of P_0. These terms result from the secular terms of the higher orders.

It is to be noted that the values P_0 and φ_0 enter into the solutions (2.10) and (2.11) in a different fashion. It is clear from Eqs. (2.4) and (2.5) that with each solution $P(t)$ and $\theta(t)$, the functions $P(t)+$const, $\theta(t)$ provide another solution. Thus, P_0 enters the solution in a simple additive way, while the dependence of the solution on φ_0 is more complicated.

It is seen that in the time interval 2π the point (P_0,φ_0) undergoes transformations of the form (2.12) and (2.13). Consider first the changes of the phase only modulo 2π; in each period 2π the phase will vary by an amount

$$\Delta\varphi_0=-\tfrac{1}{2}\pi\gamma\cos2\varphi_0+0(\gamma^2). \tag{2.14}$$

There are two distinct values of φ_0 leading to $\Delta\varphi_0=0$; namely,

$$\varphi_0'=(\pi/4)+0(\gamma)\quad\text{and}\quad\varphi_0''=(3\pi/4)+0(\gamma), \tag{2.15}$$

as easily observed from (2.14) and the requirement $\Delta\varphi_0=0$.

For $\gamma\ll2\pi$ in the neighborhood of φ_0' one has

$$\Delta\varphi_0>0\ \text{ for }\ \varphi_0>\varphi_0';\ \Delta\varphi_0<0\ \text{ for}\varphi_0<\varphi_0'. \tag{2.16}$$

For φ in the neighborhood of φ_0'' one has

$$\Delta\varphi_0<0\ \text{ for }\ \varphi_0>\varphi_0'';\ \Delta\varphi_0>0\ \text{ for }\ \varphi_0<\varphi_0''. \tag{2.17}$$

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$$\Delta\varphi_0<0\ \text{ for }\ \varphi_0>\varphi_0'';\ \Delta\varphi_0>0\ \text{ for }\ \varphi_0<\varphi_0''. \tag{2.17}$$

In other words, the point φ_0' is a point of repulsion for neighboring phases under the transformation (2.14), while the point φ_0'' is a point of attraction. This means that φ_0'' is a stable phase which any initial phase will approach eventually.

While the initial phase φ_0 approaches its stable value φ_0'', changes will occur also in the energy term P. Once the stable phase has been reached, the energy term increases in each interval 2π by a fixed positive amount,

$$\Delta P_0=+\pi\gamma+0(\gamma^2), \tag{2.18}$$

so that the energy ultimately grows beyond any bound as was actually observed by Mandelstam and Papalexi in their experiments with a linear parametric generator.

3. FREQUENCIES OTHER THAN $\omega=2$

All experiments evidence so far available relates to the case when $\omega=2$ in the d.e. (2.2) and (2.3). In comparison, t:stio to whether p.e. may occur for other valω hzen neglected. It can be shown, however, at leas far as the terms of the first order are rnece p.e. does not appear for any frequency tha= 2.

Proceec presly one finds

$$\theta_1 = -(1/\omega t - \text{l}(\omega+2)]\cos 2\varphi_0 \sin(\omega+2)t$$
$$-[1/4(]\cos\sin(\omega-2)t$$
$$+[1/4(]\sin\cos(\omega+2)t-1]$$
$$-4(\omega]\sin 2\varphi_0[\cos(\omega-2)t-1]. \quad (3.1)$$

It is seet if ?, θ_1 is a periodic function with period 2π is ier) containing harmonics. It is to be notat third term on the right side of Eq. (3.1)e lirω=2 becomes the secular term appearing]. (:. In fact, it is sufficient to set $\omega-2=x$ to] to the limit $(\sin xt/x)_{x\to 0}=t$, which givt teLikewise, one finds that the last term in (s zet the limit $\omega=2$. In a similar manner ccert: that the expression for $P_1(t)$ does not a se terms; namely,

$$P_1(t) = -\sin 2\varphi_0\{[1/2(\omega+2)]\sin(\omega+2)t + [1/2(\omega-2)]\sin(\omega-2)t\}$$
$$-\cos 2\varphi_0\{[1/2(\omega+2)][\cos(\omega+2)t-1] - [1/2(\omega-2)][\cos(\omega-2)t-1]\}, \quad (3.2)$$

inasmuch as all terms are periodic if $\omega\neq 2$. One finds that the second term on the right of Eq. (3.2) degenerates into a secular term at the limit $\omega=2$ and that P_2 does not contain secular terms, while θ_2 does. This means that p.e. exists only in still higher orders and for that reason is negligible.

4. NONLINEAR CASE

The great majority of cases encountered in applications are of the nonlinear type. The Melde-Rayleigh experiment is an example of this type. In fact, the stationary amplitude in this case is reached owing to the nonlinear elasticity of the vibrating wire for larger amplitudes. Moreover, in all cases there is a dissipation of energy in one form or another.

The perturbation procedure outlined in the preceding sections gives a convenient tool for the investigation of the nonlinear range of the phenomenon of p.e. to which the classical methods of d.e. with periodic coefficients do not apply. For that reason, instead of Eq. (2.1) we shall investigate now a d.e. of the form,

$$\ddot{x}+p\dot{x}+f(x)+\gamma x\cos 2t=0,$$

where, as is frequently done in applications, the nonlinear term $f(x)$ may be assumed to be of the form $x+\epsilon x^3$; that is, we will consider the equation

$$\ddot{x}+p\dot{x}+(1+\gamma\cos 2t)x+\epsilon x^3=0. \quad (4.1)$$

Moreover, we will assume that ϵ and p are small numbers of the same order as γ. The form (4.1) results from keeping only the terms of the first order and assuming $\omega=2$ as previously.

The perturbation procedure yields the following differential equations:

$$dP_1/dt = -2B\sin^2\theta_0 - \cos 2t\sin 2\theta_0 - \tfrac{1}{2}A\rho_0(\sin 2\theta_0 + \tfrac{1}{2}\sin 4\theta_0), \quad (4.2)$$

$$d\theta_1/dt = -\tfrac{1}{2}B\sin 2\theta_0 - \cos 2t\cos^2\theta_0 - A\rho\cos^4\theta_0, \quad (4.3)$$

where $A=\epsilon/\gamma$, $B=p/\gamma$, and $\theta_0=\varphi_0-t$ as before.

The integration yields

$$P_1 = -\tfrac{1}{2}(\sin 2\varphi_0+2B)t + \text{periodic terms}, \quad (4.4)$$

$$\theta_1 = -\tfrac{1}{4}(\cos 2\varphi_0+\tfrac{3}{2}A\rho_0)t + \text{periodic terms}. \quad (4.5)$$

After one period 2π the quantities P_1 and θ_1 undergo

$$\Delta P = -\pi\gamma(\sin 2\varphi_0+2B)+0(\gamma^2);$$
$$\Delta\varphi = -\tfrac{1}{2}\pi\gamma(\cos 2\varphi_0+\tfrac{3}{2}A\rho_0)+0(\gamma^2). \quad (4.6)$$

Setting $2\pi\gamma=\Delta t$ and considering Δt as dt in the problem involving the study of motion in the course of many periods 2π, one can replace the difference equations (4.6) by the following differential equations describing the behavior of the system in the long run

$$d\rho/dt = -\tfrac{1}{2}\rho(\sin 2\varphi+2B), \quad (4.7)$$

$$d\varphi/dt = -\tfrac{1}{4}(\cos 2\varphi+\tfrac{3}{2}A\rho). \quad (4.8)$$

The singular point of this system is given by equations,

$$\sin 2\varphi_0 = -2B; \quad \cos 2\varphi_0 = -\tfrac{3}{2}A\rho_0,$$

which gives

$$\rho_0 = (\tfrac{2}{3}A)(1-4B^2)^{\frac{1}{2}}. \quad (4.9)$$

In order to investigate the stability of the singular point, we form the variational equations of Eqs. (4.7) and (4.8), viz.:

$$d\delta\rho/dt = -\rho_0\cos 2\varphi_0\cdot\delta\varphi; \quad d\delta\varphi/dt = -\tfrac{3}{8}A\,\delta\rho + \tfrac{1}{2}\sin 2\varphi_0\delta\varphi. \quad (4.10)$$

The characteristic equation of (4.10) is

$$S^2 - \tfrac{1}{2}\sin 2\varphi_0 S - \tfrac{3}{8}A\rho_0\cos 2\varphi_0 = 0. \quad (4.11)$$

Replacing the coordinates of the singular point ρ_0, φ_0 by their values gives

$$S^2+BS+\tfrac{1}{4}(1-4B^2)=0. \quad (4.11a)$$

The singular point is a saddle point if $1-4B^2<0$, that is, if $\gamma<2p$. If $1-4B^2>0$, one has either a nodal point if $\gamma<(5)^{\frac{1}{2}}p$, or a focal point if $\gamma>(5)^{\frac{1}{2}}p$, these singularities being stable.

From Eq. (4.9) one notes that the condition of reality of the stationary amplitude ρ_0 is $\gamma>2p$, that is, the same as for the absence of the saddle point, which is obvious, since no closed periodic trajectory can exist enclosing a singular point with index $j=-1$.

For a conservative system ($B=0$), Eq. (4.11a) becomes $S^2+\frac{1}{4}=0$, that is, the singular point is a vortex

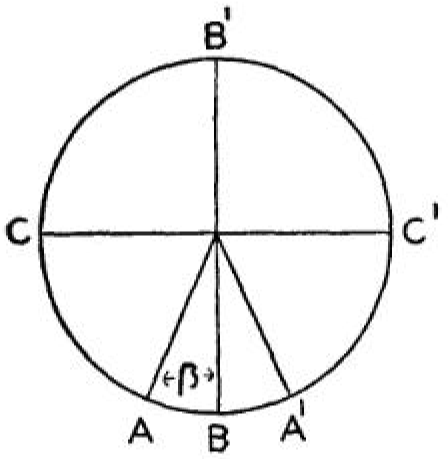

FIG. 1. Sector of self-excitation.

point. In this case from Eq. (4.9)

$$\rho_0 = 2\gamma/3\epsilon. \tag{4.12}$$

We conclude, therefore, that a conservative nonlinear system excited parametrically is generally unstable in the sense in which the word "stability" is understood in application. In other words, if a perturbation deviates the system from its stationary condition, this effect will not be reduced in time, since the representative point in the (ρ, φ) plane will describe indefinitely a closed curve containing the vortex point in its interior. In the (ρ, θ) plane this circumstance results in parasitic amplitude and phase modulations. Since the disturbances are erratic, so will be these modulations, although on *the average* there will be still a stability (in the sense of Liapounoff) in that the motion will remain in a bounded region of the phase plane. From this standpoint the presence of a dissipation of energy is beneficial in that it smoothes out these parasitic phenomena.

5. CONDITIONS OF SELF-EXCITATION OF NONLINEAR SYSTEMS

In the case of linear systems (2.4), (2.5) it was possible to eliminate the variable ρ on the right-hand side of (2.4) by introducing an auxiliary variable $P(t)$ and to thus reduce the system to a relatively simple form. For a nonlinear system this is not the case, and it is necessary to limit the analysis to a qualitative discussion. A difficulty presents itself if one attempts to set $\rho=0$ in order to determine the conditions of self-excitation from rest. Since ρ is the energy content of the system, it is clear that the condition $\rho=0$ entails also that all other dynamical variables (currents, voltages, velocities, etc.) are also zero, in which case the concept of the *phase* φ loses its physical significance.

In reality there always exists a small initial energy content in any physical system. Thus, for instance, if the system is electrical, stray charges on a condenser account for this initial electrostatic energy ρ_1. As to the phase φ, it is to be assumed as entirely arbitrary. For that reason it is necessary to investigate the starting of the phenomenon for *any value* of the initial phase φ_1.

From Eq. (4.7) it follows that $d\rho/dt>0$ if $(\sin 2\varphi + 2B)<0$. Assuming first $0<2B<1$, this condition is fulfilled if $\sin 2\varphi<0$ and $|\sin 2\varphi|>2B$. This defines a sector AOA' (Fig. 1) which we may call the *sector of self-excitation* (s.s.e. for short). If 2φ is in s.s.e., ρ increases. If 2φ is outside this sector, the small initial ρ_1 will be further reduced. The semi-angle β determining the s.s.e. is given by the relation $\cos\beta = 2B$.

For reasons which will appear later, we shall consider the boundaries OA and OA' of the s.s.e. as belonging to it. For a nondissipative system the s.s.e. becomes the lower half-circle CBC'. When $\gamma=2p$ (which is the threshold between the region of saddle points and that of nodal points), the s.s.e. shrinks to the line OB.

We shall investigate four different cases when the initial phase $2\varphi_0$ is situated in the sectors BOA', AOB, $B'OA'$, and $B'OA$.

In BOA' $\cos 2\varphi_1>0$ and, therefore, $d\varphi/dt<0$. The phase angle will move clockwise and will enter the sector AOB. Since in this sector $\cos 2\varphi<0$, the two terms on the right side of Eq. (4.8) are of opposite signs, so that $d\varphi/dt=0$ when $|\cos 2\varphi| = \frac{3}{2}A\rho$. In AOB ρ increases while the phase, for which $d\varphi/dt=0$, shifts gradually toward OA. During this shift $d\rho/dt$ decreases and becomes zero at OA, where $d\varphi/dt$ also becomes zero by Eq. (4.8). The phase angle $2\varphi_0$ of the singular point is thus OA.

If the initial phase $2\varphi_1$ is in $B'OA'$, $d\varphi/dt<0$ and, since in this sector $d\rho/dt<0$, the small initial value ρ_1 is still further reduced, and we can neglect the second term on the right of Eq. (4.8). Under these conditions, the phase $2\varphi_0$ will enter the s.s.e. through the boundary OA' and will reach the stable phase OA in the manner which we have just investigated.

If $2\varphi_1$ is in $B'OA$, $d\varphi/dt>0$ so that the phase will enter the s.s.e. through its boundary OA. This time OA will not be a position of equilibrium for 2φ on account of the fact that $\rho\cong 0$ during this passage through OA. Therefore, the phase 2φ will enter AOB, where ρ will increase to a value for which $d\varphi/dt=0$. Since in AOB ρ is increasing, this position of equilibrium for ρ will gradually shift itself toward OA, where both positions of equilibrium for ρ and for φ will be simultaneously attained as was previously shown.

Summing up, from any initial phase $2\varphi_1$, the ultimate stable phase $2\varphi_0$ will be OA; and when the phase settles on this value, the variable ρ also reaches a stable equilibrium which corresponds thus to the singular point of the system (4.7) and (4.8).

It remains to investigate the case when $B>\frac{1}{2}$ (i.e., when $\gamma<2p$). From Eq. (4.7) it follows that $d\rho/dt<0$ for any 2φ. Since ρ, originally small, decreases continuously, from Eq. (4.8) it follows that $\cos 2\varphi$ determines the stability of the phase. By a simple argument one finds that $2\varphi_0=\pi/2$ is an unstable phase and $2\varphi_0 = 3\pi/2$ is a stable phase. This case corresponds at the same time to the condition for the existence of a saddle point.

We are thus led to identify the existence of a saddle